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Simultaneous Chebychev Approximation of a Set of Bounded Complex-Valued Functions

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INTRODUCTION

For N a fixed positive integer, we denote by A a compact metric space which contains at least N distinct points; the symbol $|x-y|$ will denote the distance between two points, $x, y \in A$. For every bounded complex-valued function, g, defined on A, the norm of g is given by $\|g\| = \sup_{x\in A} |g(x)|$ (where $|g(x)|$ denotes the absolute value of the complex number $g(x)$). For M a positive real number, we denote by $F(= F(M))$ a nonempty class of complex-valued functions defined on A, such that if $f \in F$, then $\|f\| \leqslant M$. Further, we let $q_k(x)$ $(k = 1,\ldots,N)$ be a Chebychev system of continuous complex-valued functions defined on A, i.e., for any choice of complex numbers $\lambda_1, \ldots, \lambda_N$ $(\sum_{k=1}^N |\lambda_k| > 0)$, the function $\sum_{k=1}^N \lambda_k q_k(x)$ vanishes at at most $N-1$ distinct points of A. This means that, given N distinct points $x_i \in A$ $(1 \leqslant i \leqslant N)$, and N complex numbers z_i $(1 \leqslant i \leqslant N)$, there exists a unique set of complex numbers λ_k $(1 \leqslant k \leqslant N)$ such that the function $\sum_{k=1}^N \lambda_k q_k(x)$ takes on the value z_i at x_i $(1 \leqslant i \leqslant N)$; i.e., $\sum_{k=1}^N \lambda_k q_k(x_i) = z_i$ $(1 \leqslant i \leqslant N)$ (see, e.g., [*1*, p. 24]). We denote by P the class of all linear combinations of $q_1, \ldots, q_N$, i.e., P consists of exactly those functions which are of the form $\sum_{k=1}^N \lambda_k q_k(x)$, $x \in A$, λ_k complex numbers $(1 \leqslant k \leqslant N)$.

The purpose of this paper is to investigate the uniqueness of an element $q \in P$ which satisfies the equation

$$\inf_{p\in P} \sup_{f\in F} \|p - f\| = \sup_{f\in F} \|q - f\|.$$

We think of q as being an element of P which best approximates the family F. Special cases of this problem were investigated by Tonelli [*2*] and later by Kolmogorov [*3*]. In Kolmogorov's problem, F consisted of one continuous complex-valued function, defined on a compact set which contained at least $N+1$ distinct points. And in Tonelli's problem, F consisted of one continuous complex-valued function, defined on a compact subset of the complex plane, with $q_k(x) = x^{k-1}$ $(1 \leqslant k \leqslant N)$, x complex. More recently, Dunham [*4*] studied the problem, under the assumption that P was a family of real-valued functions, unisolvent of degree N, on a compact interval of the real line. He considered the cases: (i) F consists of one bounded real-valued function, (ii) F consists of an upper semicontinuous real-valued function, f^+, and a lower semicontinuous real-valued function, f^-, with $f^+ \geqslant f^-$ pointwise, and (iii) F consists of a finite number of continuous real-valued functions.

In Section 1 we treat the problem of existence of an element of P which best approximates F. In Section 2 we state a uniqueness theorem, which is the main theorem of the paper. The approach we have taken hinges on Lemma 1.3. The idea expressed in this lemma was contained in a private communication to J. B. Diaz, from P. Frederickson, dated September 1, 1968. Finally, in Section 3 we investigate special cases of the theorems of Section 2.

SECTION 1

THEOREM 1.1. *There exists an element $q \in P$ such that*

$$\inf_{p\in P} \sup_{f\in F} \|p-f\| = \sup_{f\in F} \|q-f\|.$$

Proof. Since $\|f\| \leqslant M$, one has $\inf_{p\in P} \sup_{f\in F} \|p-f\| < \infty$. Let $\langle p_n \rangle$ be a sequence in P such that

$$\lim_{n\to\infty} [\sup_{f\in F} \|p_n - f\| - \inf_{p\in P} \sup_{f\in F} \|p-f\|] = 0.$$

For each n and every $f \in F$ one has

$$\|p_n\| \leqslant \|p_n - f\| + \|f\| \leqslant \sup_{f\in F} \|p_n - f\| + M$$
$$= \inf_{p\in P} \sup_{f\in F} \|p-f\| + [\sup_{f\in F} \|p_n - f\| - \inf_{p\in P} \sup_{f\in F} \|p-f\|] + M.$$

Since the term in brackets tends to zero as n tends to infinity, the sequence $\langle p_n \rangle$ is uniformly bounded. Thus, $\langle p_n \rangle$ contains a subsequence which converges to an element of P (see, e.g., [*1*, p. 16]). Without loss, we assume that there exists an element $q \in P$ such that $\lim_{n\to\infty} \|p_n - q\| = 0$. Further, for each n,

$$0 \leqslant \sup_{f\in F} \|q-f\| - \inf_{p\in P} \sup_{f\in F} \|p-f\|$$
$$\leqslant \sup_{f\in F} [\|p_n - f\| + \|q - p_n\|] - \inf_{p\in P} \sup_{f\in F} \|p-f\|$$
$$= [\sup_{f\in F} \|p_n - f\| - \inf_{p\in P} \sup_{f\in F} \|p-f\|] + \|q-p_n\|.$$

Since the term in brackets tends to zero as n approaches infinity, and since $\lim_{n\to\infty} \|q-p_n\| = 0$, one concludes that

$$\sup_{f\in F} \|q-f\| = \inf_{p\in P} \sup_{f\in F} \|p-f\|.$$

This completes the proof of the theorem.

Next, we make two definitions. Letting C denote the complex plane, we define the set-valued functions $h(x)$ and $h^*(x)$ by

$$h(x) = \{z \in C \mid f(x) = z, f \in F\}, \qquad x \in A,$$

and

$$h^*(x) = \bigcap_{\epsilon > 0} \left(\bigcup_{|x-y|<\epsilon} h(y) \right)^0, \qquad x \in A,$$

where the superscript, 0, denotes closure in C. (As defined above, h^* is an upper semicontinuous set-valued function; see, e.g., [5, p. 148].)

The next two lemmas are used repeatedly in what follows.

LEMMA 1.1. *Let $x \in A$. Then $z \in h^*(x)$ if and only if there exists a sequence of ordered pairs $\langle (x_n, z_n) \rangle$ such that* (1) $\langle x_n \rangle \subset A$, (2) $\lim_{n\to\infty} x_n = x$, (3) $z_n \in h(x_n)$ $(n = 1, 2, \ldots)$, *and* (4) $\lim_{n\to\infty} z_n = z$.

Proof. Suppose, first, that a sequence $\langle (x_n, z_n) \rangle$ satisfying (1)–(4) exists. Then for $\epsilon > 0$ there exists a positive integer N such that, for $n > N$, one has

$$\text{(i)} \quad |x - x_n| < \epsilon \qquad \text{and} \qquad \text{(ii)} \quad z_n \in h(x_n) \subset \bigcup_{|x-y|<\epsilon} h(y).$$

Thus, since $\lim_{n\to\infty} z_n = z$, one has

$$z \in \left(\bigcup_{|x-y|<\epsilon} h(y) \right)^0.$$

But $\epsilon > 0$ was arbitrarily chosen, which means that

$$z \in \bigcap_{\epsilon > 0} \left(\bigcup_{|x-y|<\epsilon} h(y) \right)^0;$$

i.e., $z \in h^*(x)$.

Conversely, if

$$z \in \bigcap_{\epsilon > 0} \left(\bigcup_{|x-y|<\epsilon} h(y) \right)^0, \qquad \text{then } z \in \left(\bigcup_{|x-y|<\epsilon} h(y) \right)^0$$

for all $\epsilon > 0$. In particular, for each positive integer n,

$$z \in \left(\bigcup_{|x-y|<1/n} h(y) \right)^0.$$

Thus, there exists an ordered pair (x_n, z_n) such that $x_n \in A$, $|x - x_n| < 1/n$, $z_n \in h(x_n)$, and $|z - z_n| < 1/n$. Hence, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} z_n = z$. This completes the proof of Lemma 1.1.

LEMMA 1.2. *For each* $x \in A$, $h^*(x) = [h^*(x)]^*$; *i.e.*,

$$\bigcap_{\epsilon>0}\left(\bigcup_{|x-y|<\epsilon} h(y)\right)^0 = \bigcap_{\epsilon>0}\left(\bigcup_{|x-y|<\epsilon} h^*(y)\right)^0, \qquad x \in A.$$

Proof. Since $h(y) \subset h^*(y)$ for all $y \in A$, it follows that for all $\epsilon > 0$, one has

$$\left(\bigcup_{|x-y|<\epsilon} h(y)\right)^0 \subset \left(\bigcup_{|x-y|<\epsilon} h^*(y)\right)^0, \qquad x \in A,$$

and hence

$$\bigcap_{\epsilon>0}\left(\bigcup_{|x-y|<\epsilon} h(y)\right)^0 \subset \bigcap_{\epsilon>0}\left(\bigcup_{|x-y|<\epsilon} h^*(y)\right)^0, \qquad x \in A.$$

It remains to show that the inclusion sign can be reversed. Let $x \in A$ and let

$$z \in \bigcap_{\epsilon>0}\left(\bigcup_{|x-y|<\epsilon} h^*(y)\right)^0.$$

To show that

$$z \in \bigcap_{\epsilon>0}\left(\bigcup_{|x-y|<\epsilon} h(y)\right)^0,$$

it suffices, by Lemma 1.1, to show that there exists a sequence of ordered pairs $\langle(x_n, z_n)\rangle$ such that (1) $\langle x_n\rangle \subset A$, (2) $\lim_{n\to\infty} x_n = x$, (3) $z_n \in h(x_n)$ $(n = 1,2,\ldots)$, and (4) $\lim_{n\to\infty} z_n = z$.

By Lemma 1.1 there exists a sequence of ordered pairs $\langle(y_n, \xi_n)\rangle$ such that (1) $\langle y_n\rangle \subset A$, (2) $\lim_{n\to\infty} y_n = x$, (3) $\xi_n \in h^*(y_n)$ $(n = 1,2,\ldots)$, and (4) $\lim_{n\to\infty} \xi_n = z$. Since $\xi_n \in h^*(y_n)$ $(n = 1,2,\ldots)$, it follows by Lemma 1.1 again that there exists a sequence of ordered pairs $\langle(x_{nj}, z_{nj})\rangle_j$ $(n = 1,2,\ldots)$ such that (1) $\langle x_{nj}\rangle_j \subset A$, (2) $\lim_{j\to\infty} x_{nj} = y_n$ $(n = 1,2,\ldots)$, (3) $z_{nj} \in h(x_{nj})$ $(n, j = 1,2,\ldots)$, and (4) $\lim_{j\to\infty} z_{nj} = \xi_n$ $(n = 1,2,\ldots)$. Without loss (by choosing a subsequence and relabeling, if necessary) we can assume that $|y_n - x_{nn}| < 1/n$ and $|\xi_n - z_{nn}| < 1/n$ $(n = 1,2,\ldots)$. Now we define a new sequence of ordered pairs, by $(x_n, z_n) = (x_{nn}, z_{nn})$ $(n = 1,2,\ldots)$. Then, $\langle x_n\rangle \subset A$, $z_n \in h(x_n)$ $(n = 1,2,\ldots)$, and since

$$|x - x_n| \leqslant |x - y_n| + |y_n - x_n|,$$

and

$$|z - z_n| \leqslant |z - \xi_n| + |\xi_n - z_n| \qquad (n = 1,2,\ldots),$$

it follows that

$$\lim_{n\to\infty} x_n = x,$$

and

$$\lim_{n\to\infty} z_n = z.$$

This completes the proof of Lemma 1.2.

Remark. We note that if A and C are metric spaces and F is a family of functions from A into the set of all subsets of C, then both Lemmas 1.1 and 1.2 remain valid.

LEMMA 1.3. *If* $p \in P$, *then*

$$\sup_{f\in F} \|p-f\| = \sup_{x\in A} \sup_{z\in h^*(x)} |p(x)-z|.$$

Proof. Since $h^*(x) \supset h(x)$, $x \in A$, it follows that for each $x \in A$ and each $f \in F$, one has

$$|p(x)-f(x)| \leqslant \sup_{z\in h(x)} |p(x)-z| \leqslant \sup_{z\in h^*(x)} |p(x)-z|$$
$$\leqslant \sup_{x\in A} \sup_{z\in h^*(x)} |p(x)-z|.$$

Thus,

$$\sup_{f\in F} \|p-f\| = \sup_{f\in F} \sup_{x\in A} |p(x)-f(x)|$$
$$\leqslant \sup_{x\in A} \sup_{z\in h^*(x)} |p(x)-z|.$$

It remains to show that the inequality can be reversed. We choose a sequence of ordered pairs $\langle (x_n, z_n)\rangle$ such that $x_n \in A$, $z_n \in h^*(x_n)$ $(n = 1,2,\ldots)$ and such that

$$\lim_{n\to\infty} |p(x_n)-z_n| = \sup_{x\in A} \sup_{z\in h^*(x)} |p(x)-z|.$$

Since $z_n \in h^*(x_n)$ $(n = 1,2,\ldots)$, it follows from Lemma 1.1 that there exists a sequence of ordered pairs (η_n, ξ_n) such that (1) $\langle\eta_n\rangle \subset A$, (2) $|x_n - \eta_n| < 1/n$, (3) $\xi_n \in h(\eta_n)$, and (4) $|z_n - \xi_n| < 1/n$ $(n = 1,2,\ldots)$. Since $\xi_n \in h(\eta_n)$ $(n = 1,2,\ldots)$, there exists an element of F, call it f_n, such that $f_n(\eta_n) = \xi_n$ $(n = 1,2,\ldots)$. Thus, for $n = 1, 2, \ldots$, one has

$$\sup_{x\in A} |p(x)-f_n(x)| \geqslant |p(\eta_n)-f_n(\eta_n)| = |p(\eta_n)-\xi_n|$$
$$\geqslant |p(\eta_n)-z_n| - |\xi_n - z_n|$$
$$\geqslant |p(x_n)-z_n| - |p(x_n)-p(\eta_n)| - |\xi_n - z_n|,$$

and hence

$$\sup_{f\in F}\|p-f\| = \sup_{f\in F}\sup_{x\in A}|p(x)-f(x)|$$
$$\geqslant \overline{\lim_{n\to\infty}}\sup_{x\in A}|p(x)-f_n(x)|$$
$$\geqslant \overline{\lim_{n\to\infty}}\,[|p(x_n)-z_n|-|p(x_n)-p(\eta_n)|-|\xi_n-z_n|].$$

Using the facts that p is uniformly continuous on A, $\lim_{n\to\infty}|x_n-\eta_n|=0$ and $\lim_{n\to\infty}|\xi_n-z_n|=0$, one has that

$$\overline{\lim_{n\to\infty}}\,[|p(x_n)-z_n|-|p(x_n)-p(\eta_n)|-|\xi_n-z_n|]$$
$$=\lim_{n\to\infty}|p(x_n)-z_n| = \sup_{x\in A}\sup_{z\in h^*(x)}|p(x)-z|.$$

This completes the proof of Lemma 1.3.

Remark. We note that Lemma 1.3 remains valid under the assumptions that A and C are metric spaces, F is a family of functions from A into the set of all subsets of C, and $p(x)$ is a uniformly continuous function from A into C. Under these hypotheses, one must admit the possibility that the conclusion of the lemma takes the form $\infty=\infty$.

For $p\in P$, define a set $D_p\subset A\times C$ by

$$D_p=\{(x,z)\in A\times C\,|\,z\in h^*(x)\quad\text{and}\quad |p(x)-z|=\sup_{f\in F}\|p-f\|\}.$$

Thus, the set D_p may depend upon the choice of $p\in P$. The next lemma asserts that for each $p\in P$, the corresponding set D_p is nonempty.

LEMMA 1.4. *Let $p\in P$, and let D_p be the corresponding set in $A\times C$, as defined above. Then $D_p\neq\varnothing$.*

Proof. Let $x\in A$. Since $h^*(x)$ is compact, there exists a point $\bar z(x)\in h^*(x)$ such that

$$\sup_{z\in h^*(x)}|p(x)-z| = |p(x)-\bar z(x)|.$$

Let $\langle x_n\rangle$ be a sequence in A such that

$$\lim_{n\to\infty}|p(x_n)-\bar z(x_n)| = \sup_{x\in A}|p(x)-\bar z(x)|.$$

Since A is compact, we can assume without loss that $\lim_{n\to\infty}x_n=x_0\in A$. Since $\bar z(x_n)$ is a bounded sequence in C, we can further assume without loss that $\lim_{n\to\infty} \bar{z}(x_n) = z_0 \in C$. From Lemmas 1.1 and 1.2, one concludes that $z_0 \in [h^*(x_0)]^* = h^*(x_0)$. Further,

$$\begin{aligned} 0 &\leqslant \sup_{x\in A} \sup_{z\in h^*(x)} |p(x) - z| - |p(x_0) - z_0| \\ &= \sup_{x\in A} |p(x) - \bar{z}(x)| - |p(x_0) - z_0| \\ &= \lim_{n\to\infty} [|p(x_n) - \bar{z}(x_n)| - |p(x_0) - z_0|] \\ &\leqslant \overline{\lim_{n\to\infty}} |(p(x_n) - p(x_0)) + (z_0 - \bar{z}(x_n))| \\ &\leqslant \lim_{n\to\infty} |p(x_n) - p(x_0)| + \lim_{n\to\infty} |z_0 - \bar{z}(x_n)| = 0 \end{aligned}$$

(where we have used the continuity of p). One concludes that

$$\sup_{x\in A} \sup_{z\in h^*(x)} |p(x) - z| = |p(x_0) - z_0|.$$

Using Lemma 1.3, it follows that

$$|p(x_0) - z_0| = \sup_{x\in A} \sup_{z\in h^*(x)} |p(x) - z| = \sup_{f\in F} \|p - f\|;$$

i.e., $(x_0, z_0) \in D_p$. This completes the proof of Lemma 1.4.

SECTION 2

Theorem 2.1 gives a characterization of an element of P which best approximates F.

THEOREM 2.1. *If $q \in P$ is such that*

$$\inf_{p\in P} \sup_{f\in F} \|p - f\| = \sup_{f\in F} \|q - f\| = E,$$

then for every $p \in P$ there exists an ordered pair (x_0, z_0) (possibly depending on p), $x_0 \in A$, $z_0 \in h^*(x_0)$, such that $|q(x_0) - z_0| = E$ and $\mathrm{Re}\{(q(x_0) - z_0)\overline{p(x_0)}\} \geqslant 0$, where the overbar denotes complex conjugate.

Proof. Let $D_q = \{(x,z) \in A \times C \,|\, z \in h^*(x)$ and $|q(x) - z| = E\}$. Lemma 1.4 asserts that $D_q \neq \varnothing$. We assume that the theorem is false; that is, that there exists a $p \in P$ such that for every $(x,z) \in D_q$ one has

$$\mathrm{Re}\{(q(x) - z)\overline{p(x)}\} < 0.$$

(Clearly $p(x) \not\equiv 0$.) We show, first, that there exists a positive number ϵ such that for all $(x,z) \in D_q$, one actually has

$$\mathrm{Re}\{(q(x) - z)\overline{p(x)}\} \leqslant -2\epsilon < 0.$$

Let $\langle(x_n, z_n)\rangle$ be a sequence in D_q such that

$$\lim_{n\to\infty} \operatorname{Re}\{(q(x_n) - z_n)\overline{p(x_n)}\} = \sup_{(x,z)\in D_q} \operatorname{Re}\{(q(x) - z)\overline{p(x)}\}.$$

Since A is compact, we can assume without loss that $\lim_{n\to\infty} x_n = \eta \in A$. Since F is a uniformly bounded family of functions, the sequence $\langle z_n\rangle$ is bounded, and hence we can further assume without loss that $\lim_{n\to\infty} z_n = \xi \in C$. Thus, by Lemmas 1.1 and 1.2, one has $\xi \in [h^*(\eta)]^* = h^*(\eta)$. Further, since for $\delta > 0$ and n sufficiently large,

$$\begin{aligned} 0 &\leqslant \big||q(x_n) - z_n| - |q(\eta) - \xi|\big| \\ &\leqslant \overline{\lim_{n\to\infty}} |q(x_n) - z_n - q(\eta) + \xi| + \delta \\ &\leqslant \delta + \lim_{n\to\infty} |q(x_n) - q(\eta)| + \lim_{n\to\infty} |\xi - z_n| = \delta, \end{aligned}$$

one has that

$$E = \lim_{n\to\infty} |q(x_n) - z_n| = |q(\eta) - \xi|.$$

Thus, $(\eta, \xi) \in D_q$. Since

$$\lim_{n\to\infty} \operatorname{Re}\{(q(x_n) - z_n)\overline{p(x_n)}\} = \operatorname{Re}\{(q(\eta) - \xi)\overline{p(\eta)}\},$$

it suffices to define ϵ by $\epsilon = -\frac{1}{2}\operatorname{Re}\{(q(\eta) - \xi)\overline{p(\eta)}\}$.

We now show that for λ (> 0) sufficiently small, one has

$$\sup_{f\in F} \|(q + \lambda p) - f\| < \sup_{f\in F} \|q - f\| = E,$$

which contradicts the definition of q; i.e., the inequality contradicts the fact that q is the best approximation to F. By Lemma 1.3, one has

$$\sup_{f\in F} \|(q + \lambda p) - f\| = \sup_{x\in A} \sup_{z\in h^*(x)} |(q(x) + \lambda p(x)) - z|,$$

so it suffices to show that if $\lambda > 0$ is small enough, then

$$\sup_{x\in A} \sup_{z\in h^*(x)} |(q(x) + \lambda p(x)) - z| < E.$$

We argue, first, that there exists an open set $G \subset A \times C$ such that $G \supset D_q$, and such that if $(x, z) \in G$, $x \in A$, $z \in h^*(x)$, then

$$\operatorname{Re}\{(q(x) - z)\overline{p(x)}\} < -\epsilon.$$

Since $\operatorname{Re}\{(q(x) - z)\overline{p(x)}\}$ is a continuous real-valued function on $A \times C$, it suffices to let

$$G = \{(x, z) \in A \times C \mid \operatorname{Re}\{(q(x) - z)\overline{p(x)}\} < -\epsilon\}.$$

Clearly, G is an open set in $A \times C$ (G is an inverse image of the set of real numbers $\{y \in R | y < -\epsilon\}$) and $G \supset D_q$.

Now, if $B = \max_{x \in A} |p(x)|$ (> 0), and if $0 < \lambda < \epsilon/B^2$, then for $(x,z) \in G$, $x \in A$, $z \in h^*(x)$, one has

$$\begin{aligned} |(q(x) + \lambda p(x)) - z|^2 &= |q(x) - z|^2 + 2\lambda \operatorname{Re}\{(q(x) - z)\overline{p(x)}\} + \lambda^2 |p(x)|^2 \\ &\leqslant E^2 - 2\lambda\epsilon + \lambda^2 B^2 \\ &= E^2 - \lambda(\epsilon + (\epsilon - \lambda B^2)) < E^2 - \lambda\epsilon. \end{aligned}$$

(In particular, $0 < E^2 - \lambda\epsilon$, a fact which will be used later.)

Now let G^c denote the complement, in $A \times C$, of the set G. We show that there exists a positive number, δ, such that if $(x,z) \in G^c$, $x \in A$, $z \in h^*(x)$, then

$$|q(x) - z| < E - \delta.$$

Lemma 1.3 ensures that $|q(x) - z| \leqslant E$, for all pairs (x,z), $x \in A$, $z \in h^*(x)$. Thus, if there exists no such δ, then there exists a sequence $\langle (x_n, z_n) \rangle \subset G^c$, $x_n \in A$, $z_n \in h^*(x_n)$ ($n = 1,2,\ldots$) such that $\lim_{n\to\infty} |q(x_n) - z_n| = E$. Since A is compact, we can assume without loss, that $\lim_{n\to\infty} x_n = \eta_1 \in A$. Further, since $\langle z_n \rangle$ is a bounded sequence in C, we can assume without loss, that $\lim_{n\to\infty} z_n = \xi_1 \in C$. By Lemmas 1.1 and 1.2, $\xi_1 \in [h^*(\eta_1)]^* = h^*(\eta_1)$. Thus, by a continuity argument used above,

$$E = \lim_{n\to\infty} |q(x_n) - z_n| = |q(\eta_1) - \xi_1|,$$

and hence $(\eta_1, \xi_1) \in D_q$. But this contradicts the fact that G^c is a closed set whose complement contains D_q. Thus, there exists a $\delta > 0$ such that if $(x,z) \in G^c$, $x \in A$, $z \in h^*(x)$, then

$$|q(x) - z| < E - \delta.$$

And hence, if $0 < \lambda < \delta/2B$, then

$$\begin{aligned} |(q(x) + \lambda p(x)) - z| &\leqslant |q(x) - z| + \lambda |p(x)| \\ &< E - \delta + \lambda B \\ &< E - \frac{\delta}{2}. \end{aligned}$$

We have shown that for $x \in A$, $z \in h^*(x)$, $0 < \lambda < \min\{\epsilon/B^2, \delta/2B\}$, one has

$$|(q(x) + \lambda p(x)) - z| < \max\left\{(E^2 - \lambda\epsilon)^{1/2}, E - \frac{\delta}{2}\right\} < E.$$

Hence,

$$\sup_{x\in A} \sup_{z\in h^*(x)} |(q(x) + \lambda p(x)) - z| < E.$$

This completes the proof of Theorem 2.1.

THEOREM 2.2. *Let $q \in P$ be such that*

$$\inf_{p \in P} \sup_{f \in F} \|p - f\| = \sup_{f \in F} \|q - f\| = E,$$

and define the set $D_q \subset A \times C$ by

$$D_q = \{(x,z) \in A \times C \,|\, z \in h^*(x) \text{ and } |q(x) - z| = E\}.$$

If for every two points in D_q of the form (x,z) and (x,z'), one has

$$\mathrm{Re}\{(q(x) - z)\overline{(q(x) - z')}\} > 0,$$

then q is unique; i.e., if $q_1 \in P$ and $\sup_{f \in F} \|q_1 - f\| = E$, then $q_1 = q$.

(The condition $\mathrm{Re}\{(q(x) - z)\overline{(q(x) - z')}\} > 0$ can be interpreted geometrically to mean that "the angle" between the two straight lines determined by the pairs $(q(x), z)$ and $(q(x), z')$ is, in absolute value, less than $\pi/2$.)

Proof. If $E = 0$, then $\|q - f\| = 0$ for every $f \in F$; but this is possible if and only if F consists of exactly one element, f, and $f = q$. In this case, q is trivially unique. In what follows, we assume $E > 0$.

We begin by showing that the number of points $(x,z) \in D_q$ which have distinct first coordinates is at least N. Assuming that this is not the case, we let x_i $(i = 1, \ldots, m < N)$ be those distinct points of A for which there exist $z_i \in h^*(x_i)$ $(i = 1, \ldots, m)$ such that $(x_i, z_i) \in D_q$. Let $p \in P$ be such that $p(x_i) = -(q(x_i) - z_i)$, where z_i is an element of $h^*(x_i)$ chosen arbitrarily, but such that $(x_i, z_i) \in D_q$ $(i = 1, \ldots, m)$. Then for $i = 1, \ldots, m$, one has

$$\mathrm{Re}\{(q(x_i) - z_i)\overline{p(x_i)}\} = -|q(x_i) - z_i|^2 = -E^2 < 0.$$

If for some i, $1 \leqslant i \leqslant m$, there exist two points z_i and z_i', both belonging to $h^*(x_i)$, such that both (x_i, z_i) and (x_i, z_i') belong to D_q, then the hypothesis of the theorem ensures that

$$\mathrm{Re}\{(q(x_i) - z_i')\overline{p(x_i)}\} = -\mathrm{Re}\{(q(x_i) - z_i')\overline{(q(x_i) - z_i)}\} < 0.$$

Thus, under the assumption that $m < N$, there exists an element $p \in P$ such that

$$\mathrm{Re}\{(q(x) - z)\overline{p(x)}\} < 0$$

for all $(x,z) \in D_q$, which violates the conclusion of Theorem 2.1. One concludes that $m \geqslant N$.

Now we assume that for some $q_1 \in P$, one has

$$\sup_{f \in F} \|q_1 - f\| = E.$$

Then, for all $f \in F$ one has

$$\|\tfrac{1}{2}(q+q_1)-f\| \leqslant \tfrac{1}{2}\|q-f\| + \tfrac{1}{2}\|q_1-f\| = E,$$

and hence

$$\sup_{f\in F}\|\tfrac{1}{2}(q+q_1)-f\| \leqslant E.$$

On the other hand, from the definition of E, one has

$$\sup_{f\in F}\|\tfrac{1}{2}(q+q_1)-f\| \geqslant E.$$

Thus

$$\sup_{f\in F}\|\tfrac{1}{2}(q+q_1)-f\| = E.$$

By the above argument, there exist N distinct points $x_i \in A$ $(i=1,\ldots,N)$ and corresponding points $z_i \in h^*(x_i)$ $(i=1,\ldots,N)$, such that

$$\begin{aligned}|\tfrac{1}{2}(q(x_i)+q_1(x_i))-z_i| &= |\tfrac{1}{2}(q(x_i)-z_i)+\tfrac{1}{2}(q_1(x_i)-z_i)|\\ &= E \qquad (i=1,\ldots,N).\end{aligned}$$

But since

$$\begin{aligned}|\tfrac{1}{2}(q(x_i)-z_i)+\tfrac{1}{2}(q_1(x_i)-z_i)| &\leqslant \tfrac{1}{2}|q(x_i)-z_i| + \tfrac{1}{2}|q_1(x_i)-z_i|\\ &\leqslant \tfrac{1}{2}E+\tfrac{1}{2}E = E \qquad (i=1,\ldots,N),\end{aligned}$$

one must have

(i) $$|q(x_i)-z_i| = |q_1(x_i)-z_i| = E \qquad (i=1,\ldots,N),$$

and

(ii) $$|\tfrac{1}{2}(q(x_i)-z_i)+\tfrac{1}{2}(q_1(x_i)-z_i)| = \tfrac{1}{2}|q(x_i)-z_i| + \tfrac{1}{2}|q_1(x_i)-z_i| \qquad (i=1,\ldots,N).$$

Equations (i) and (ii) hold if and only if

$$q(x_i)-z_i = q_1(x_i)-z_i \qquad (i=1,\ldots,N).$$

Thus, q and q_1 agree on N distinct points of A, which means $q=q_1$. This completes the proof of Theorem 2.2.

Remark. If A consists of at least $N+1$ points, then the argument above can be used to show that the number of points $(x,z) \in D_q$ with distinct first coordinates is at least $N+1$.

Theorem 2.1 has a converse which was not needed for the proof of the uniqueness theorem but is given below for completeness.

THEOREM 2.3. *If $q \in P$ is such that for every $p \in P$ there exists an ordered pair (which may depend upon p) (x_0,z_0), $x_0 \in A$, $z_0 \in h^*(x)$ with the property that*

$$|q(x_0)-z_0| = \sup_{f\in F}\|q-f\|$$

and

$$\operatorname{Re}\{(q(x_0)-z_0)\overline{p(x_0)}\}\geqslant 0,$$

then

$$\inf_{p\in P}\sup_{f\in F}\|p-f\|=\sup_{f\in F}\|q-f\|.$$

Proof. Let $p\in P$ and choose (x_0,z_0) such that

$$x_0\in A,\qquad z_0\in h^*(x_0),\qquad |q(x_0)-z_0|=\sup_{f\in F}\|q-f\|,$$

and

$$\operatorname{Re}\{(q(x_0)-z_0)\overline{(p(x_0)-q(x_0))}\}\geqslant 0.$$

Then, using Lemma 1.3, one obtains

$$\begin{aligned}\sup_{f\in F}\|p-f\|&=\sup_{f\in F}\|(q-f)+(p-q)\|\\&=\sup_{x\in A}\sup_{z\in h^*(x)}|(q(x)-z)+(p(x)-q(x))|\\&\geqslant|(q(x_0)-z_0)+(p(x_0)-q(x_0))|\\&=[|q(x_0)-z_0|^2+2\operatorname{Re}\{(q(x_0)-z_0)\overline{(p(x_0)-q(x_0))}\\&\qquad+|p(x_0)-q(x_0)|^2]^{1/2}\\&\geqslant|q(x_0)-z_0|=\sup_{f\in F}\|q-f\|.\end{aligned}$$

Thus,

$$\inf_{p\in P}\sup_{f\in F}\|p-f\|=\sup_{f\in F}\|q-f\|,$$

which completes the proof.

SECTION 3

In this section, we examine special cases of the approximation problem treated in Sections 1 and 2.

Case 1. In the event that F consists of one continuous complex-valued function, f, one has

$$f(x)\equiv h(x)\equiv h^*(x).$$

Theorems 2.1–2.3, under the assumption that A consists of at least $N+1$ points, reduce to theorems of Kolmogorov [*3*]. In particular, the approximating function q of Theorem 2.2, is unique.

Case 2. In the event that F consists of a finite number of continuous complex-valued functions $f_1,\ldots,f_m$, one has $h^*(x)=h(x)$, $x\in A$.

Case 3. In the event that

(1) F is a non-empty family of uniformly bounded real-valued functions,

(2) $q_k(x)$ $(1 \leqslant k \leqslant N)$ is a Chebychev system of continuous real-valued functions,

(3) P consists of all functions of the form $\sum_{k=1}^{N} \lambda_k q_k$, λ_k real numbers $(1 \leqslant k \leqslant N)$,

Theorems 2.1–2.3 remain valid.† Under the assumptions (1)–(3), the condition

$$\operatorname{Re}\{(q(x)-z)\overline{(q(x)-z')}\} > 0,$$

of Theorem 2.2 reduces to

$$(q(x)-z)(q(x)-z') > 0,$$

which means that x is not a straddle point, as defined in [*4*]. (One actually has $(q(x)-z)(q(x)-z') < 0$, for every two points in D_q of the form (x,z), (x,z'), $z \neq z'$.)

It seems worthwhile to give slightly different versions of Theorems 2.1–2.3, under the assumptions (1)–(3) of Case 3. To do this, we define two functions, $F^+(x)$ and $F^-(x)$, by

$$F^+(x) = \inf_{\delta>0} \sup_{0\leqslant|x-y|<\delta} \sup_{f\in F} f(y), \qquad x \in A,$$

and

$$F^-(x) = \sup_{\delta>0} \inf_{0\leqslant|x-y|<\delta} \inf_{f\in F} f(y), \qquad x \in A.$$

The function F^+ is upper semicontinuous and the function F^- is lower semicontinuous. The ideas of Theorems 2.1 and 2.3 can be combined, to give the following

THEOREM 3.1. *Let F, $q_k(x)$ $(1 \leqslant k \leqslant N)$, and P be as in* (1)–(3) *of Case* 3, *and let $q \in P$. A necessary and sufficient condition that*

$$\inf_{p\in P} \sup_{f\in F} \|p-f\| = \sup_{f\in F} \|q-f\|,$$

is that, for $p \in P$, there exists an $x_0 \in A$ $(x_0 = x_0(p))$ such that either

$$|q(x_0) - F^+(x_0)| = \sup_{f\in F} \|q-f\|,$$

and

$$(q(x_0) - F^+(x_0))p(x_0) \geqslant 0,$$

or

$$|q(x_0) - F^-(x_0)| = \sup_{f\in F} \|q-f\|,$$

† These "real versions" of Theorems 2.1–2.3 do not appear to follow immediately as special cases of these theorems.

and

$$(q(x_0) - F^{-}(x_0))p(x_0) \geqslant 0.$$

The proof of this theorem can be modeled after the proofs of Theorems 2.1 and 2.3, by first using the Corollary of [6] in place of Lemma 1.3. An analogue of Theorem 2.2 is given next.

THEOREM 3.2. *Let* F, q_k $(1 \leqslant k \leqslant N)$, *and* P *be as in* (1)–(3) *of Case* 3, *and let* $q \in P$ *be such that*

$$\inf_{p \in P} \sup_{f \in F} \|p - f\| = \sup_{f \in F} \|q - f\|.$$

If, for every $x \in A$, *one has*

$$[q(x) - F^{+}(x)]\,[q(x) - F^{-}(x)] \neq -(\sup_{f \in F} \|q - f\|)^2,$$

then q *is unique; i.e., if there exists a* $q' \in P$ *such that*

$$\sup_{f \in F} \|q' - f\| = \inf_{p \in P} \sup_{f \in F} \|p - f\|,$$

then

$$q' = q.$$

When A is a compact interval of the real line and F consists of exactly two functions, one an upper semicontinuous function, f^{+}, and one a lower semicontinuous function, f^{-}, with $f^{+}(x) \geqslant f^{-}(x)$, $x \in A$, Theorem 3.2 is a special case of Theorems 1 and 2 of [4].

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